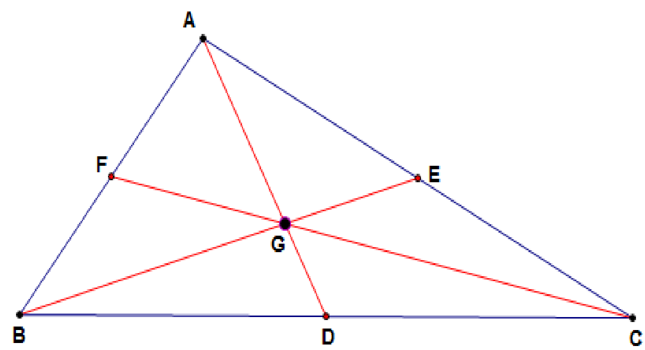
Ceva's Theorem

Now we will explore couple of theorems involving our definitions and explorations. The first of which is the Ceva's theorem.

Ceva's Theorem:

Centroid: A median of a triangle is the line segment from vertex of the triangle to the midpoint of the side opposite of that vertex. Because there are three vertices, there are of course three possible medians. One of the interesting facts about the medians of triangles is that no matter what shape the triangle, all three medians always intersect at a single point inside the triangle. This point is called the Centroid, denoted by G, of the triangle.



Centroid is also known as the center of mass of triangle. Use the exploration in part five for details.

Examine the Centroid in GSP sketch. (Click on *script view* option of GSP to get details on construction).

Ceva's Theorem: In any triangle ΔABC, the cevians AD, BE, CF are concurrent if and only if

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1 \text{ (in simple form)}$$

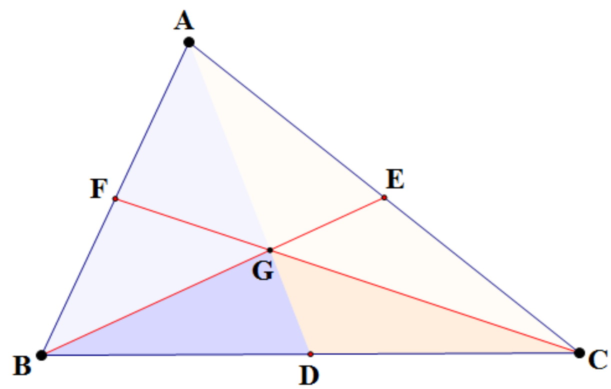
or

$$\frac{\sin BAD}{\sin DAC} \cdot \frac{\sin ABE}{\sin EBC} \cdot \frac{\sin BCF}{\sin FCA} = 1 \text{ (in trigonometric form)}$$

Note: Cevian is the line segment that connects a vertex of a triangle with the opposite side. And when three or more lines all pass through a common point, is called concurrent.

Proof: The proof of Ceva's Theorem is based on the area of triangle.

Lemma: The areas of triangles with equal altitude are proportional to the bases of the triangles.



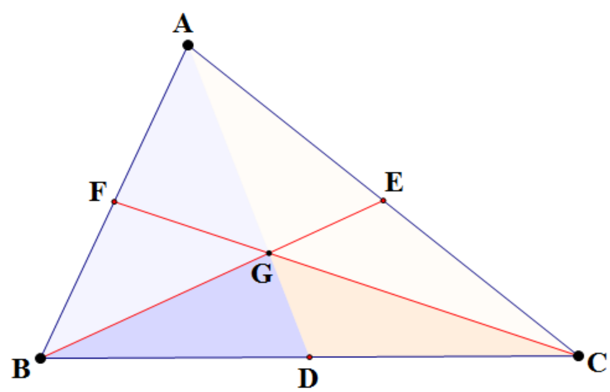
Note: (ABC) denotes the area of ΔABC.

Let AD, BE, and CF concur at point G. So we have: $\frac{BD}{DC} = \frac{(BDA)}{(CDA)} = \frac{(BDG)}{(CDG)}$

$$\frac{BD}{DC} = \frac{(BDA)-(BDG)}{(CDA)-(CDG)} = \frac{(ABG)}{(ACG)}$$

Similarly, we can get $\frac{CE}{CA} = \frac{(BCG)}{(BAG)}$ and $\frac{AF}{FB} = \frac{(CAG)}{(CBG)}$

So, $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{(CAG)}{(CBG)} \cdot \frac{(ABG)}{(ACG)} \cdot \frac{(BCG)}{(BAG)} = 1.$



For the converse, suppose $\frac{AF}{FB} = \frac{BD}{DC} = \frac{CE}{EA} = 1$ and G be the point of intersection of AD and BE. Let CG meet AB at $\bar{F}$. Then by forward argument we have

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{A\bar{F}}{\bar{F}B} = 1$$

And hence we have

$$\frac{AF}{FB} = \frac{A\bar{F}}{\bar{F}B}$$

So that both F and $\bar{F}$ divide AB in the same ratio and must therefore be the same point. Hence the theorem is proved.